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VERTEX-TO-EDGE DETOUR DISTANCE IN GRAPHS

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Abstract

In this paper, we introduce the vertex-to-edge $u-e$ path, the vertexto-edge detour distance $D(u,e)$, the vertex-to-edge $u-e$ detour, the vertex-to-edge detour eccentricity $e_{D1}(u)$, the vertex-to-edge detour radius R_1, and the vertex-to-edge detour diameter D_1 of a connected graph G, where u is a vertex and e an edge in G. We determine these parameters for some standard graphs. It is shown that $R_1 \leq D_1 \leq 2R_1+1$ for every connected graph G and that every two positive integers a and b with $a \leq b \leq 2a+1$ are realizable as the vertex-to-edge detour radius and the vertex-to-edge detour diameter, respectively, of some connected graph. Also it is shown that for any two positive integers a, b with $a \leq b$ are realizable as the vertex-to-edge radius and the vertex-to-edge detour radius, respectively, of some connected graph and also it is shown that for any two positive integers a, b with $a \leq b$ are realizable as the vertex-to-edge diameter and the vertex-to-edge detour diameter, respectively, of some connected graph. Also we introduce the vertex-to-edge detour center $C_1(G)$ and the vertex to-edge detour periphery $P_1(G)$. It is shown that the vertex-to-edge detour center of every connected graph lies in a single block. Also it is shown that every graph is the vertex-to-edge detour center of some connected graph.

***Key words*:** vertex-to-edge detour distance, vertex-to-edge detour radius, vertex-to-edge detour diameter.

***AMS Subject Classification*:** 05C12.

1 Introduction

By a graph G = (V,E) we mean a finite undirected connected simple graph. For basic graph theoretic terminologies, we refer to Chartrand and Zhang [4]. For example if one is locating an emergency facility like police station, fire station, hospital, school, college, library, ambulance depot, emergency care center, etc., then the primary aim is to minimize the distance between the facility and the location of a possible emergency. In 1964, Hakimi [6] considered the facility location problems as vertex-to-vertex distance in graphs. For any two vertices u and v in a connected graph G, the distance d(u, v) is the length of a shortest $u - v$ path in G. For a vertex v in G, the eccentricity e(v) of v is the distance between v and a vertex farthest from v in G. The minimum eccentricity among the vertices of G is its radius and the maximum eccentricity is its diameter, denoted by rad(G) and diam(G) respectively. A vertex v in G is a central vertex if e(v) = rad(G) and the subgraph induced by the central vertices of G is the center Cen(G) of G. A vertex v in G is a peripheral vertex if e(v) = diam(G) and the subgraph induced by the peripheral vertices of G is the periphery Per(G) of G. If every vertex of G is a central vertex then G is called self-centered graph.

For example if one is making an election canvass or circular bus service the distance from the location is to be maximized. In 2005, Chartrand et.al. [3] introduced and studied the concepts of detour distance in graphs. For any two vertices u and v in a connected graph G, the detour distance D(u, v) is the length of a longest $u - v$ path in G. For a vertex v in G, the detour eccentricity $e_D(v)$ of v is the detour distance between v and a vertex farthest from v in G. The minimum detour eccentricity among the vertices of G is its detour radius and the maximum detour eccentricity is its detour diameter, denoted by $rad_D(G)$ and $diam_D(G)$ respectively. Detour center, detour self-centered and detour periphery of a graph are defined similar to the center, self-centered and periphery respectively of a graph, respectively.

For example when a railway line, pipe line or highway is constructed, the distance between the respective structure and each of the communities to be served is to be minimized. In a social network an edge represents two individuals having a common interest. Thus the centrality with respect to edges have intresting application in social networks. In 2010, Santhakumaran [9] introduced the facility locational problem as vertex-to-edge distance in graphs as follows: For a vertex u and an edge e in a connected graph G, the vertex-to-edge distance is defined by $d(u,e) = \min\{d(u,v) : v \in e\}$. The vertex-to-edge eccentricity of u is defined by $e_1(u) = \max\{d(u,e) : e \in E\}$. An edge e of G such that $e_1(u) = d(u, e)$ is called a vertex-to-edge eccentric edge of u. The vertex-to-edge radius r_1 of G is defined by $r_1 = \min\{e_1(v) : v \in V\}$ and the vertex-to-edge diameter d_1 of G is defined by $d_1 = \max\{e_1(v) : v \in V\}$. A vertex v for which $e_1(v)$ is minimum is called a vertex-to-edge central vertex of G and the set of all vertex-to-edge central vertices of G is the vertex-to-edge center $C_1(G)$ of G. A vertex v for which $e_1(v)$ is maximum is called a vertex-to-edge peripheral vertex of G and the set of all vertex-to-edge peripheral vertices of G is the vertex-to-edge periphery $C_1(G)$ of G. If every vertex of G is a vertex-to-edge central vertex then G is called vertex-to-edge self-centered graph.

These motivated us to introduce a distance called the vertex-to-edge deotur distance in graphs and investigate certain results related to vertex-to-edge detour distance and other distances in graphs. For example when a dam, river and channel is constructed, the distance between the respective structure and each of the communities to be served is to be maximized. Further these ideas have intresting applications in channel assignment problem in radio technologies. Also there are useful applications of these concepts to security based communication network design. Throughout this paper, G denotes a connected graph with at least two vertices.

2 Vertex-To-Edge Detour Distance

Definition 2.1. *Let* u *be a vertex and* e *an edge in a connected graph* G. *A vertex-to-edge* u−e *path* P *is a* u−v *path, where* v *is a vertex in* e *such that* P *contains no vertices of* e *other than* v. *The vertex-to-edge detour distance* D(u,e) *is the length of a longest* u−e *path. A* u−e *path of length* D(u,e) *is called a vertex-to-edge* u−e *detour or simply* u−e *detour. For our convenience a* u−e *path of length* d(u,e) *is called a vertex-to-edge* u−e *geodesic or simply* u−e *geodesic.*

Example 2.2. *Consider the graph* G *given in Fig 2.1. For the vertex* u *and the edge* e = {v,w}, *the paths* P_1 : u,w, P_2 : u, z, r, v *and* P_3 :u, t, s, x, y, z, r, v *are* u − e *paths, while the paths* Q_1 : u,w, v *and* Q_2 : u, z, r, v,w *are not* u − e *paths. Now the vertex-to-edge distance* d(u, e) = 1 *and the vertex-to-edge detour distance* D(u, e) = 7. *Thus the vertex-to-edge detour distance is different from the vertex-to-edge distance. Also* P_3 *is a* u−e *detour and* P_1 *is a* u−e *geodesic. Note that the* v−e *and* w−e *paths are trivial.*

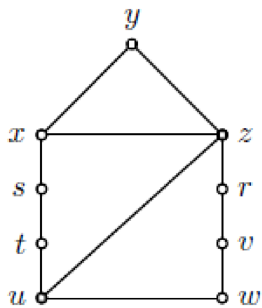

Fig. 2.1: G

Since the length of a u − e path between a vertex u and an edge e in a graph G of order n is at most n − 2, we have the following theorem.

Theorem 2.3. *For any vertex* u *and an edge* e *in a non-trivial connected graph* G *of order* n, $0 \leq d(u, e) \leq D(u, e) \leq n - 2$.

Remark 2.4. *The bounds in the Theorem 2.3 are sharp. For every vertex* u *in* G, d(u, e) = D(u, e) = 0 *if and only if* $u \in e$ *and if* G *is a path* $P : u = u_1, u_2, \dots, u_{n-1}, u_n$ *of order* n, *then* d(u, e) = D(u, e) = n−2, *where* $e = \{u_{n-1}, u_n\}$. *Also we note that if* G *is a tree, then* d(u, e) = D(u, e) *and if* G *is an even cycle with* $u \notin e$, *then* d(u, e) < D(u, e) *for every vertex* u *in* G.

Since a vertex of degree n − 1 in a graph G of order n, belongs to every edge e in G, we have the following theorem.

Theorem 2.5. *Let* G *be a connected graph of order* n *and* e *an edge in* G. *If* u *is a vertex of degree* n − 1, *then* D(u, e) = 0.

The converse of the Theorem 2.5 is not true. For the graph G given in Fig. 2.1, D(u, e) = 0, where e = {u, z}, but $\deg(u) \neq n - 1$.

Theorem 2.6. *Let* $K_{n,m}$ (n < m) *be a complete bipartite graph with the partition* V_1, V_2 *of* $V(K_{n,m})$ *such that* $|V_1| = n$ *and* $|V_2| = m$. *Let* u *be a vertex and* e *an edge such that* $u \notin e$ *in* $K_{n,m}$, *then* $D(u,C) = \begin{cases} 2n - 2 \text{ if } u \in V_1 \\ 2n - 1 \text{ if } u \in V_2 \end{cases}$

Proof. Let $V_1 = \{x_1, x_2, \dots, x_n\}$ and $V_2 = \{y_1, y_2, \dots, y_n, y_{n+1}, \dots, y_m\}$. Without loss of generality assume that $e = \{x_n, y_n\}$ is an edge and $u = x_1$ or $u = y_1$.

Case 1. $u = x_1$. Let $P_1 : u = x_1, y_1, x_2, y_2, \dots, x_{n-1}, y_{n-1}, x_n$ be a longest u − e path, which has n vertices of V_1 and n − 1 vertices of V_2. It must contain 2n − 1 vertices of $K_{n,m}$. So that its length is 2n − 2. Also $P_2 : u = x_1, y_1, x_2, y_2, \dots, x_{n-1}, y_n$ be a longest u−e path, which has n−1 vertices of V_1 and n−1 vertices of V_2. It must contain 2n − 2 vertices of $K_{n,m}$. So that its length is 2n − 3. Thus D(u,e) = 2n − 2 if $u \in V_1$.

Case 2. $u = y_1$. Let $Q_1 : u = y_1, x_1, y_2, x_2, \dots, y_{n-1}, x_{n-1}, y_{n+1}, x_n$ be a longest u − e path, which has n vertices of V_1 and n vertices of V_2. It must contain 2n vertices of $K_{n,m}$. So that its length is 2n − 1. Also $Q_2 : u = y_1, x_1, y_2, x_2, \dots, y_{n-1}, x_{n-1}, y_n$ be a longest u−e path, which has n−1 vertices of V_1 and n vertices of V_2. It must contain 2n − 1 vertices of $K_{n,m}$. So that its length is 2n − 2. Thus D(u, e) = 2n − 1 if $u \in$ V2.

Corollary 2.7. *Let* u *be a vertex and* e *an edge in a complete bipartite graph* $K_{n,n}$ *such that* $u \notin e$, *then* $D(u, e) = 2n - 2$.

Since every tree has unique $u - e$ path between a vertex u and an edge e, we have the following theorem.

Theorem 2.8. *If* G *is a tree, then* $d(u, e) = D(u, e)$ *for every vertex* u *and an edge* e *in* G.

The converse of the Theorem 2.8 is not true. For every vertex u in K_3 with $u \notin e$, $d(u, e) = D(u, e) = 1$ and for every vertex u in K_3 with $u \in e$, $d(u, e) = D(u, e) = 0$.

Definition 2.9. *For a vertex* u *and an edge* e *in a connected graph* G, *a* $u - e$ *path* P *is said to be* $u - e$ *hamiltonian path if* $P \cup \{e\}$ *contains every vertex of* G.

Theorem 2.10. *For a vertex* u *and an edge* e *in a connected graph* G *of order* $n \geq 2$, *there exists an integer* k *such that* $D(u,e)=k$ *if and only if* G *is hamiltonian-connected and* $k= n-2$.

Proof. If G is a hamiltonian-connected graph of order $n \geq 2$, then there exists a $u-e$ hamiltonian path between a vertex u and an edge e in G and so $D(u, e) = n-2$. Conversely, assume to the contrary, that there exists a connected graph G of order $n \geq 2$ such that $D(u, e) = k$ for a vertex u and an edge e in G, but $k < n - 2$. Let $uv, xv \in E$. Since $D(u, e) = k$, there exists a $u-e$ detour P of length k in G. Then $P \cup \{xv\} \cup \{uv\}$, forms a cycle C_{k+2} of length k+2 in G. Since $n > k + 2$ and G is connected, there exists a vertex $y \in V(G) - V(C_{k+2})$ such that y is adjacent to some vertex z in C_{k+2}. Assume that $C_{k+2} : z = v_1, v_2, \ldots, v_{k+2}, v_1 = z$. However then $y, z = v_1, v_2, \ldots, v_{k+1}, v_{k+2}$ is an $y - e'$ path of length $k + 2$, where $e' = \{v_{k+1}, v_{k+2}\}$ and so $D(u, e') \geq k + 2$, which is a contradiction.

3 Vertex-To-Edge Detour Center

Definition 3.1. *The vertex-to-edge detour eccentricity* $e_{D1}(u)$ *of a vertex* u *in a connected graph* G *is defined as* $e_{D1}(u) = \max \{D(u, e) : e \in E\}$. *An edge* e *for* *which* $e_{D1}(u) = D(u, e)$ *is called a vertex-to-edge detour eccentric edge of* u. *The vertex-to-edge detour radius of* G *is defined as,* $R_1 = rad_{D1}(G) = \min \{e_{D1}(v) : v \in V\}$ *and the vertex-to-edge detour diameter of* G *is defined as,* $D_1 = diam_{D1}(G) = \max \{e_{D1}(v) : v \in V\}$. *A vertex* v *in* G *is called a vertex-to-edge detour central vertex if* $e_{D1}(v) = R_1$ *and the vertex-to-edge detour center of* G *is defined as,* $C_{D1}(G) = Cen_{D1}(G) = \langle\{v \in V : e_{D1}(v) = R_1\}\rangle$. *A vertex* v *in* G *is called a vertex-to-edge detour peripheralvertex if* $e_{D1}(v) = D_1$ *and the vertex-to-edge detour periphery of* G *is defined as,* $P_{D1}(G) = Per_{D1}(G) = \langle\{v \in V : e_{D1}(v) = D_1\}\rangle$. *If every vertex of* G *is a vertex-to-edge detour central vertex, then* G *is called a vertex-to-edge detour self centered graph. If* G *is a vertex-to-edge detour self-centered graph, then* G *is its own vertex-to-edge detour periphery.*

Example 3.2. *For the graph* G *given in Fig. 3.1, the set of all edges in* G *are given by,* E $=\{e_1=\{v_1,v_2\}, e_2=\{v_1,v_3\}, e_3=\{v_2,v_3\}, e_4=\{v_3,v_3\}, e_5=\{v_4,v_5\}, e_6=\{v_5,v_6\}, e_7=\{v_6,v_7\}, e_8=\{v_4,v_7\}, e_9=\{v_7,v_8\}, e_{10}=\{v_8,v_{10}\}, e_{11}=\{v_4,v_{10}\}, e_{12}=\{v_4,v_9\}, e_{13}=\{v_9,v_{10}\}, e_{14}=\{v_{10}, v_{14}\}, e_{15}=\{v_{13},v_{14}\}, e_{16}=\{v_{12},v_{13}\}, e_{17}=\{v_{11},v_{12}\}, e_{18}=\{v_{10},v_{11}\}, e_{19}=\{v_{11}, v_{14}\}, e_{20}=\{v_{10}, v_{11}\}, e_{21}=\{v_{10}, v_{13}\}, e_{22}=\{v_{10}, v_{12}\}, e_{23}=\{v_{12}, v_{14}\}\}$.

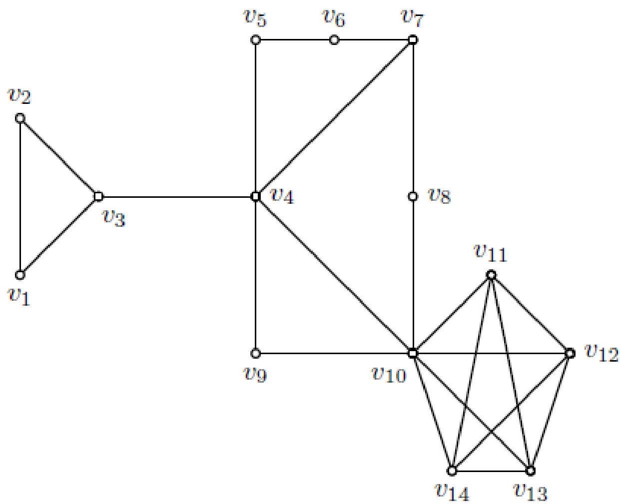

Fig. 3.1: G

The vertex-to-edge eccentricity $e_1(v)$, *the vertex-to-edge detour eccentricity* $e_{D1}(v)$ *of all the vertices of* G *are given in Table 1.*

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}
$e_1(v)$	4	4	3	2	3	4	3	3	3	3	4	4	4	4
$e_{D_1}(v)$	11	11	9	8	8	8	8	9	9	7	11	11	11	11

Table 1

The vertex-to-edge detour eccentric edge of all the vertices of G *are given in Table 2.*

Vertex v	*Vertex-to-Edge Detour Eccentric Edge e*
$v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_9$	$e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{23}$
v_5	$e_1, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{23}$
$v_{10}, v_{11}, v_{12}, v_{13}, v_{14}$	e_1

Table 2

The vertex-to-edge radius $r_1 = 2$, *the vertex-to-edge diameter* $d_1 = 4$, *the vertex-to-edge detour radius* $R_1 = 7$ *and the vertex-to-edge detour diameter* $D_1 = 11$. *Also the vertex-to-edge center* $C_1(G) = \langle\{v_4\}\rangle$, *the vertex-to-edge periphery* $P1(G) = \langle\{v_1, v_2, v_6, v_{11}, v_{12}, v_{13}, v_{14}\}\rangle$, *the vertex-to-edge detour center* $C_{D1}(G) = \langle\{v_{10}\}\rangle$ *and the vertex-to-edge detour periphery* $P_{D1}(G) = \langle\{v_1, v_2, v_{11}, v_{12}, v_{13}, v_{14}\}\rangle$.

The vertex-to-edge detour radius R_1 and the vertex-to-edge detour diameter D_1 of some standard graphs are given in Table 3.

G	K_n	P_n	$C_n (n \geq 4)$	$W_n (n \geq 5)$	$K_{n,m} (m \geq n)$
R_1	$n-2$	$\lfloor \frac{n-2}{2} \rfloor$	$n-2$	$n-2$	$2(n-1)$
D_1	$n-2$	$n-2$	$n-2$	$n-2$	$\begin{cases} 2(n-1), & \text{if } n=m \\ 2n-1 & \text{if } n>m \end{cases}$

Table 3

Remark 3.3. *In a connected graph* G, $C_1(G)$, $C_{D1}(G)$ *and* $P_1(G)$, $P_{D1}(G)$ *need not be same. For the the graph G given in Fig 3.1,* $C_1(G)$, $C_{D1}(G)$ *and* $P_1(G)$, $P_{D1}(G)$ *are distinct.*

Remark 3.4. *In a connected graph* G, $C_{D1}(G)$ *and* $P_{D1}(G)$ *need not be connected. For the graph* G *given in Fig 3.2,* $C_{D1}(G) = \langle\{v_2, v_4\}\rangle$ *and* $P_{D1}(G) = \langle\{v_1, v_3, v_5\}\rangle$ *are disconnected.*

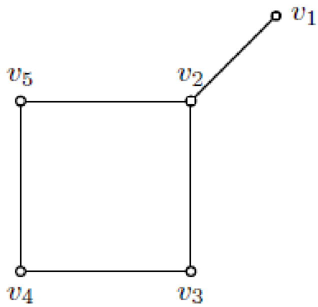

Fig. 3.2: G

Example 3.5. *The complete graph* K_n, *the cycle* C_n, *the wheel* W_n *and the complete bipartite graph* $K_{n,n}$ *are vertex-to-edge detour self centered graphs.*

Remark 3.6. *A vertex-to-edge self-centered graph need not be a vertex-to-edge detour self centered graph. Let* $K_{n,m} (n < m)$ *be a complete bipartite graph with the partition* V_1, V_2 *of* $V(K_{n,m})$ *with* $|V_1| = n$ *and* $|V_2| = m$ *such that* $C_1(G) = \langle\{V(K_{n,m})\}\rangle$, $C_{D1}(G) = \langle\{V_1(K_{n,m})\}\rangle$.

Theorem 3.7. *Let* G *be a connected graph of order* n. *Then*

(i) $0 \le e_1(u) \le e_{D1}(u) \le n - 2$ *for every vertex* u *in* G.

(ii) $0 \le r_1 \le R_1 \le n - 2$.

(iii) $0 \le d_1 \le D_1 \le n - 2$.

Proof. This follows from Theorem 2.3.

Remark 3.8. *The bounds in the Theorem 3.7 (i) are sharp. If* $G = K_2$, *then* $e_1(u) = e_{D1}(u) = 0 = n - 2$ *for every vertex* u *in* G *and and if* G *is a path* $P : u_1, u_2, \ldots, u_{n-1}$, un *of order* n, *then* $e_1(u) = e_{D1}(u) = n - 2$, *where* $u = u_1$ *or* $u = u_n$. *Also we note that if* G *is a tree, then* $e_1(u) = e_{D1}(u)$ *for every vertex* u *in* G *and for the graph* G *given in Fig. 2.1,* $0 < e_1(u) < e_{D1}(u) < n - 2$.

Theorem 3.9. *For every connected graph* G, $R_1 \le D_1 \le 2R_1 + 1$.

Proof. By definition $R_1 \leq D_1$. Now let $P : u = u_1, u_2, \ldots, u_{n-1}$ be a vertex-to-edge diametral path of length D_1 connecting a vertex u and an edge e, where $e = \{u_{n-1}, u_n\}$, so that $D_1 = D(u,e) = D(u, u_{n-1})$ and let x be a vertex of G such that $e_{D1}(x) = R_1 = D(x, e) = D(x, e')$, where $e' = \{u, u_2\}$. It follows that $D_1 = D(u, e) \leq D(u, x) + D(x, e) = D(x, e') + 1 + D(x, e) \leq 2R_1 + 1$.

Remark 3.10. *The bounds in the Theorem 3.9 are sharp. Let* $K : \{u_1, u_2, u_3\}$ *and* $K' : \{v_1, v_2, v_3\}$ *be to copies of* K_3. *Let* G *be a graph obtained by identifying* u_1 *in* K *and* v_1 *in* K'. *It is easy to verify that* $R_1 = 1$ *and* $D_1 = 3$.

Ostrand [7] Showed that every two positive positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter respectively of some connected graph and Chartrand et. al. [3] showed that every two positive positive integers a and b with $a \leq b \leq 2a$ are realizable as the detour radius and detour diameter respectively of some connected graph. Now we have a realization theorem for the vertex-to-edge detour radius and the vertex-to-edge detour diameter of some connected graph.

Theorem 3.11. *For each pair* a, b *of positive integers with* $a \leq b \leq 2a + 1$, *there exists a connected graph* G *with* $R_1 = a$ *and* $D_1 = b$.

Proof. Case 1. a = b. Let $G = C_{a+2} : u_1, u_2, \ldots, u_{a+2}, u_1$ be a cycle of order a + 2. Then $e_{D1}(u_i) = a$ for $1 \leq i \leq a + 2$. It is easy to verify that every vertex x in G with $e_{D1}(x) = a$. Thus $R_1 = a$ and $D_1 = b$ as a = b.

Case 2. $2 \leq a < b \leq 2a + 1$. Let $C_{a+2} : u_1, u_2, \ldots, u_{a+2}, u_1$ be a cycle of order a + 2 and $P_{b-a+1} : v_1, v_2, \ldots, v_{b-a+1}$ be a path of order b − a + 1. We construct the graph G of order b + 2 by identifying the vertex u_1 of C_{a+2} with v_1 of P_{b-a+1}. It is easy to verify that

$e_{D1}(u_i) = a$ for i = 1

$$e_{D1}(u_i) = \begin{cases} b - i + 2 & \text{if } 2 \leq i \leq \left\lceil \frac{a+2}{2} \right\rceil \\ b - a + i - 2 & \text{if } \left\lceil \frac{a+2}{2} \right\rceil < i \leq a + 2 \end{cases}$$

$e_{D1}(v_i) = a + i - 1$ for $1 \leq i \leq b - a + 1$

Inparticular $e_{D1}(u_i) = e_{D1}(v_i) = a$ for i = 1

$e_{D1}(u_i) = b$ for $i = 2, a + 2$

$e_{D1}(v_i) = b$ for $i = b - a + 1$

It is easy to verify that there is no vertex x in G with eD1(x) < a and there is no vertex y in G with $e_{D1}(y) > b$. Thus $R_1 = a$ and $D_1 = b$ as $a < b$.

Chartrand et. al. [3] showed that every pair a, b of integers with $1 \leq a \leq b$ is realizable as the radius and the detour radius of some connected graph. Now we have a realization theorem for the vertex-to-edge radius and the vertex-to-edge detour radius of some connected graph.

Theorem 3.12. *For any two positive integers* a, b *with* $a \leq b$, *there exists a connected graph* G *such that* $r_1 = a$ *and* $R_1 = b$.

Proof. Let $K = K_{b-a+2}$ be the complete graph $V(K) = \{z_1, z_2, \ldots, z_{b-a+2}\}$. Let $P_1 : x_1, x_2, \ldots, x_{a+1}$ and $P_2 : y_1, y_2, \ldots, y_{a+1}$ be two paths of order $a + 1$. We construct the graph G of order b+ a + 2 by identifying the vertices x_1 in P_1 with z1 in K, also identifying the vertices y_1 in P_2 with z_{b-a+2} in K.

It is easy to verify that

$e_1(z_i) = a$ for $1 \leq i \leq b - a + 2$

$e_{D1}(z_i) = b$ for $1 \leq i \leq b - a + 2$

$e_1(x_i) = a + i - 1$ for $1 \leq i \leq a$

$e_{D1}(x_i) = b + i - 1$ for $1 \leq i \leq a$

It is easy to verify that there is no vertex x in G with $e_1(x) < a$ and $e_{D1}(x) < b$. Thus $r_1 = a$ and $R_1 = b$ as $a \leq b$.

Chartrand et. al. [3] showed that every pair a, b of integers with $1 \leq a \leq b$ is realizable as the diameter and the detour diameter of some connected graph. Now we have a realization theorem for the vertex-to-edge diameter and vertex-to-edge detour diameter of some connected graph.

Theorem 3.13. *For any two positive integers* a, b *with* $a \leq b$, *there exists a connected graph* G *such that* $d_1 = a$ *and* $D_1 = b$.

Proof. Let $K = K_{b-a+3}$ be the complete graph $V(K) = \{v_1,v_2, \ldots ,v_{b-a+3}\}$ and let $P : u_1, u_2,\ldots,u_a$ be a path of order a. We construct the graph G of order $b + 2$ by identifying the vertices u1 in P with v_1 in K. It is easy to verify that

$$e_1(v_i) = \begin{cases} a-2 & \text{if } i = 1 \\ a-1 & \text{if } 2 \le i \le b-a+3 \end{cases}$$

$$e_{D1}(v_i) = \begin{cases} b-a+1 & \text{if } i = 1 \\ b & \text{if } 2 \le i \le b-a+2 \end{cases}$$

$$e_1(u_i) = \begin{cases} a-i-1 & \text{if } 1 \le i \le \left\lfloor \frac{a}{2} \right\rfloor \\ i & \text{if } \left\lfloor \frac{a}{2} \right\rfloor < i \le a \end{cases}$$

$$e_{D1}(u_i) = \begin{cases} b-a+i & \text{if } 1 \le i \le a \text{ for } b-a+i \ge a-2 \\ a-2 & \text{if } 1 \le i \le a \text{ for } b-a+i \ge a-2 \end{cases}$$

It is easy to verify that there is no vertex x in G with $e_1(x) > a$ and $e_{D1}(x) > b$. Thus $d_1 = a$ and $D_1 = b$ as $a \le b$.

Harary and Norman [5] showed that the center of every connected graph G lies in a single block of G and Chartrand et. al. [3] showed that the detour center of every connected graph G lies in a single block of G. Also Santhakumaran [9] showed that the vertex-to-edge center of every connected graph G lies in a single block of G. Now we have the following theorem for the vertex-to-edge detour center of a graph.

Theorem 3.14. *The vertex-to-edge detour center of every connected graph* G *lies in a single block of* G.

Proof. Suppose that the vertex-to-edge detour center of a connected graph G lies in more than one block. Then G contains a cut vertex v such that $G - v$ has two components G_1 and G_2, each of which contains a vertex-to-edge detour central vertices of G. Let C be a vertex-to-edge detour eccentric edge of v and let P be a vertex-to-edge longest path in G. At least one of G_1 and G_2 contains no vertices of P, say G_2 contains no vertex of P. Let w be a vertex-to-edge detour central vertex in G that belongs to G_2 and let Q be a $w - v$ longest path in G. Since v is a cut vertex, P followed by Q produces a $w - e$ longest path, whose length is greater than that of P. Hence $e_{D1}(w) \ge D(w, v) + D(v, e) = D(w, v) + e_{D1}(v) > e_{D1}(v)$. Thus $e_{D1}(w) > e_{D1}(v)$. So that w is not a vertex-to-edge detour central vertex in G, which is contradiction. Hence $C_{D1}(G)$ lies within a block of G.

Corollary 3.15. *The vertex-to-edge detour center of a tree is isomorphic to either* K_1 *or* K_2.

Theorem 3.16. *No cut vertex in a connected graph* G *is a vertex-to-edge detour peripheral vertex of* G.

Hedetniemi [see [2]] showed that every graph is the center of some connected graph. Chartrand et. al [3] showed that every graph is the detour center of some connected graph. Also Santhakumaran [9] showed that the vertex set of every graph G with at least two vertices is the vertex-to-edge center of some connected graph. Now the following theorem shows which graphs are vertex-to-edge detour center of some connected graph.

Theorem 3.17. *Every graph is the vertex-to-edge detour center of some connected graph.*

Proof. Let G be a graph of order n and let $H = G + K_{n+1}$ be the join of G and K_{n+1}. Since $e_{D1}(v) = 2(n-1)$ if $v \in V(G)$ and $e_{D1}(v) = 2n-1$ if $v \in V(K_{n+1})$, it follows that G is the vertex-to-edge detour center of H.

Theorem 3.18. *If* G *is hamiltonian graph of order* n *then* G *is vertex-to-edge detour self centered graph having* $R_1 = D_1 = n - 2$.

The converse of the theorem 3.18 is not true. For example, the Petersen graph is a non-hamiltonian vertex-to-edge detour self centered graph.

Theorem 3.19. *If* G *is a vertex-to-edge detour self-centered graph of order 3 or more then* G *is 2-connected.*

Proof. Assume, to contrary, that a vertex-to-edge detour self-centered graph G of order $n \geq 3$ is not 2-connected. By definition G is a vertex-to-edge detour periphery graph, every vertex in G is a vertex-to-edge detour peripheral vertex. Since G is not a 2-connected graph, there exists a cut-vertex v in G, which is not a vertex-to-edge detour peripheral vertex of G, by Theorem 3.16, which is a contradiction.

Bielak and Syslo [1] showed that a non-trivial graph G is the periphery of some connected graph if and only if every vertex of G has eccentricity 1 or no vertex of G has eccentricity 1. Chartrand et. al [3] showed that on connected graph G of order $n \geq 3$ and radius 1 is the detour periphery of some connected graph if and only if G is hamiltonian. Now the following theorem shows which graphs are vertex-to-edge detour periphery of some connected graph.

Theorem 3.20. *A connected graph* G *of order* $n \geq 3$ *and radius 1 is the vertex-to-edge detour periphery of some connected graph if and only if* G *is hamiltonian.*

Proof. If G is hamiltonian, then G is its own vertex-to-edge detour periphery (by Theorem 3.18). For the converse, assume to the contrary, that there exists a connected graph G of order $n \geq 3$ and radius 1 that is non-hamiltonian such that G is the vertex-to-edge detour periphery of some connected graph H. Let u be a vertex in G such that $e(u) = 1$. Since v is a vertex-to-edge detour peripheral vertex of H, it follows that $D(u,e) = D_1(H)$ for some $e = \{xy\} \in E(G)$. Let P be a $v - e$ detour in H. Since v is adjacent to every vertex in G, it follows that $P \cup \{e\}$ contains every vertex of G. However then $P \cup \{e\} \cup \{uy\}$ forms a cycle C in H and every vertex of C is then vertex-to-edge detour peripheral vertex of H. So C is a subgraph of $P_{D1}(H)$. Since no vertex of $H - V(G)$ is a vertex-to-edge detour peripheral vertex of H, it follows that $V(C) = V(G)$ and so C is a hamiltonian cycle of G. This contradicts the fact that G is not hamiltonian.

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